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Weak Precompactness in the Space of Pettis Integrable Functions

G. EMMANUELE

Department of Mathematics, University of Catania, 95125 Catania, Italy

AND

K. MUSIAL

Institute of Mathematics, University of Wroclaw, 50384 Wroclaw, Poland

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INTRODUCTION

In this paper we give a necessary and sufficient condition for weak precompactness in the space of Pettis integrable functions with relatively compact range of their indefinite integrals. The main result is formulated in the spirit of previous works by J. K. Brooks and N. Dinculeanu [1, 2], where such a theorem was obtained for strongly measurable and Pettis integrable functions. Also we show that, in a sense, our result is the best possible (see Remark 2).

PRELIMINARIES

Throughout (S, Σ, μ) will be a finite complete measure space, X will be a Banach space, and B_X will denote the closed unit ball of X. The space of (μ-) Pettis integrable functions, with values into X, will be denoted by $\mathbb{P}(\mu, X)$ (we identify weakly equivalent functions). Each $f \in \mathbb{P}(\mu, X)$ determines an X-valued measure $\nu_f \colon \Sigma \to X$ given by

$$\nu_f(E) = \int_E f(s)\, d\mu.$$

The subspace of $\mathbb{P}(\mu, X)$ consisting of those functions f for which ν_f has relatively compact range is denoted by $\mathbb{P}_c(\mu. X)$. Both $\mathbb{P}(\mu, X)$ and $\mathbb{P}_c(\mu, X)$ are endowed with the norm topology given by

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$$\|f\|_{\mathbb{P}} = \sup\left\{\int_S |x^*f(s)|\, d\mu : x^* \in B_{X^*}\right\}.$$

Each $f \in \mathbb{P}(\mu, X)$ determines, also, an operator $T_f : X^* \to L^1(\mu)$ given by

$$T_f(x^*) = x^*f$$

which is known to be weak*-weakly continuous (see [3]). Denote by B the set $B_{X^*} \times B_{L^\infty(\mu)}$ endowed with the product of weak* topologies.

For each $f \in \mathbb{P}(\mu, X)$ we can define a function $Q_f : B \to \mathbb{R}$ by the equation

$$Q_f(x^*, g) = \int_S (x^*f(s))\, g(s)\, d\mu.$$

It is obvious that Q_f is separably continuous. The following result explains when Q_f is continuous.

Proposition 1. *For each $f \in \mathbb{P}(\mu, X)$, the following conditions are equivalent:*

(i) *$v_f(\Sigma)$ is norm relatively compact,*

(ii) *T_f is weak*-norm continuous,*

(iii) *Q_f is continuous.*

Proof. The equivalence of (i) and (ii) was proved by Edgar [5]. Assuming (ii) and using the separate continuity of Q_f we get at once (iii). Indeed, if (x^*_α) and x^* are in B_{X^*} and (g_α) and g are in $B_{L^\infty(\mu)}$ and $x^*_\alpha \to x^*$, $g_\alpha \to g$ in the weak* topology, then

$$\begin{aligned}
&\left|\int_S (x^*_\alpha f(s))\, g_\alpha(s)\, d\mu - \int_S (x^*f(s))\, g(s)\, d\mu\right| \\
&\quad\leqslant \left|\int_S ((x^*_\alpha - x^*)\, f(s))(g_\alpha(s) - g(s))\, d\mu\right| \\
&\qquad + \left|\int_S (x^*f(s))(g_\alpha(s) - g(s))\, d\mu\right| \\
&\qquad + \left|\int_S ((x^*_\alpha - x^*)\, f(s))g(s) d\mu\right| \\
&\quad\leqslant 3 \int_S |((x^*_\alpha - x^*)\, f(s))|\, d\mu \\
&\qquad + \left|\int_S (x^*f(s))(g_\alpha(s) - g(s))\, d\mu\right|.
\end{aligned}$$

This gives (iii). Assume, now, (iii) and take (x_α^*) and x^* in B_{X^*} such that $x_\alpha^* \to x^*$ in the weak* topology. According to the Hahn–Banach theorem for each α there exists $g_\alpha \in B_{L^\infty(\mu)}$ such that

$$\|(x_\alpha^* - x^*) f\|_{L^1(\mu)} = \left| \int_S ((x_\alpha^* - x^*) f(s))\, g_\alpha(s)\, d\mu \right|.$$

Let (h_γ) be a weak* converging subnet of (g_α). Then, by the continuity of Q_f, we have

$$\lim_\gamma \int_S ((x_\gamma^* - x^*) f(s))\, h_\gamma(s)\, d\mu = 0.$$

This means that (x_α^*) contains a subnet (x_γ^*) such that

$$\lim_\gamma \int_S |(x_\gamma^* - x^*) f(s)|\, d\mu = 0.$$

It follows, since (x_α^*) is arbitrary, that T_f is weak*-norm continuous. This completes the whole proof.

Remark 1. Let us consider the Banach space $\mathrm{ca}(\mu, X)$ of all μ-continuous X-valued measures $v: \Sigma \to X$ with bounded semivariation equipped with the semivariation norm. For each v in $\mathrm{ca}(\mu, X)$ we define an operator $T_v: X^* \to L^1(\mu)$ by putting $T_v(x^*) = f_{x^*}^v$, where $f_{x^*}^v$ is the Radon–Nikodym derivative of $x^* v$ with respect to μ. Also, we define an operator Q_v from B into $\mathbb{R}$ by putting

$$Q_v(x^*, g) = \int_S f_{x^*}^v(s)\, g(s)\, d\mu.$$

Using the same ideas involved in the proof of Proposition 1 we are able to prove the following facts

PROPOSITION 1′. *For each $v \in \mathrm{ca}(\mu, X)$, T_v is weak*-weakly continuous.*

PROPOSITION 1″. *For each $v \in \mathrm{ca}(\mu, X)$ the following facts are equivalent:*

(i) *v has relatively compact range,*

(ii) *T_v is weak*-norm continuous,*

(iii) *Q_v is continuous.*

In order to formulate the next result let us denote by $E(h/\Xi)$ the conditional expectation of $h \in L^1(\mu)$ with to respect to a σ-algebra $\Xi \subset \Sigma$. Then we have the following fact

LEMMA. *If $H \subset L^1(\mu)$ is bounded and uniformly integrable, then for each increasing sequence (Σ_n) of sub-σ-algebras of Σ and each $g \in L^\infty(\mu)$ we have*

$$\lim_n \sup_H \left| \left\langle g, E(h/\Sigma_n) - E\left(h/\sigma\left(\bigcup_n \Sigma_n\right)\right)\right\rangle \right| = 0.$$

Proof. Since $(E(g/\Sigma_n))$ is an $L^1(\mu)$-bounded martingale we have

$$\lim_n E(g/\Sigma_n) = E\left(h/\sigma\left(\bigcup_n \Sigma_n\right)\right), \qquad \mu\text{-a.e.}$$

Using the assumptions and the Egoroff theorem we easily obtain the required result.

THE MAIN RESULT

Denote by $\mathbb{B}(B)$ the space of all real valued, bounded functions on B endowed with the sup norm.

The next proposition is a simple consequence of a simple calculation and of Proposition 1. The idea of such an embedding was taken from [1].

PROPOSITION 2. *The mapping $V: \mathbb{P}(\mu, X) \to \mathbb{B}(B)$ given by*

$$V(f) = Q_f, \qquad f \in \mathbb{P}(\mu, X)$$

is a linear isometry. Its restriction to $\mathbb{P}_c(\mu, X)$ is a linear isometry of $\mathbb{P}_c(\mu, X)$ into $C(B)$.

In order to formulate the theorem we need one more notation. If $\pi = \{A_1, A_2, ..., A_n\}$ is a partition of S into sets from Σ and $f \in \mathbb{P}(\mu, X)$ then we put

$$E_\pi f = \sum_{A \in \pi} \left(\frac{1}{\mu(A)} \int_S f(s)\, d\mu\right) \chi_A$$

(with the convention $0/0 = 0$).

THEOREM 1. *For $H \subset \mathbb{P}_c(\mu, X)$ the following conditions are equivalent:*

(a) *H is weakly precompact;*

(b) (i) *for every $A \in \Sigma$ the set*

$$H(A) = \{v_f(A): f \in H\}$$

is weakly precompact in X; (ii) *for every countable* $H_0 \subset H$ *there exists a sequence* (π_n) *of finite partitions of* S *such that* $\lim_n E_{\pi_n} f = f$ *weakly in* $\mathbb{P}_c(\mu, X)$ *uniformly for* $f \in H_0$;

(c) (j) *for every* $A \in \Sigma$ *the set* $H(A)$ *is weakly precompact*; (jj) *for every* $x^* \in X^*$ *the set*

$$x^*H = \{x^*f : f \in H\}$$

is weakly precompact in $L^1(\mu)$.

Proof. The implication (a) ⇒ (c) is a simple consequence of the continuity of the mappings $f \to \nu_f(A)$ and $f \to x^*f$.

(c) ⇒ (b). Assume (c) and take a countable subset H_0 of H. Since according to [8] each $f \in \mathbb{P}_c(\mu, X)$ is weakly measurable with respect to a separable (in the Fréchet–Nikodym sense) measure space, we may assume that all $f \in H_0$ are weakly measurable with respect to a separable complete $(S, \hat{\Sigma}, \mu \mid \hat{\Sigma})$ with $\hat{\Sigma} \subset \Sigma$. But then we can find a countable algebra $\Sigma_0 \subset \hat{\Sigma}$ such that $\hat{\Sigma}$ is the μ-completion of $\sigma(\Sigma_0)$ (in general it is not true that $\hat{\Sigma}$ is a completion of $\sigma(\Sigma_0)$ with respect to $\mu \mid \sigma(\Sigma_0)$). Then, we take as (π_n) any increasing sequence of finite Σ_0-partitions of S into sets of positive measure, which is cofinal to the net of finite Σ_0-partitions of S into sets of positive measure. For each $x^* \in X^*$ the set x^*H is weakly precompact in $L^1(\mu)$ and since $L^1(\mu)$ is weakly sequentially complete, it follows from the lemma that, for each $(x^*, g) \in B$,

$$\lim_n \int_S (x^* E_{\pi_n} f(s))\, g(s)\, d\mu = \int_S (x^* f(s))\, g(s)\, d\mu$$

uniformly for $f \in H_0$. In other words, we have

$$\lim_n z^*(E_{\pi_n} f) = z^*(f)$$

for each extreme point in the unit ball of $(C(B))^*$ and this is true uniformly on H_0. Since $(E_{\pi_n} f)$ is bounded we can apply a uniform version of the Rainwater Theorem to get the weak convergence of $(E_{\pi_n} f)$ to f in $\mathbb{P}_c(\mu, X)$ uniformly on $f \in H_0$.

(b) ⇒ (a). Let us assume the validity of (b). If π is a finite decomposition of S, then (i) yields the weak precompactness of the set

$$E_\pi(H) = \{E_\pi f : f \in H\}$$

because

$$E_\pi(H) \subset \sum_{A \in \pi} \frac{1}{\mu(A)} H(A)\, \chi_A.$$

In particular, for (π_n) from (ii) the corresponding set $E_{\pi_n}(H)$ is weakly precompact, for each $n \in N$. It follows from Lemma 10 in [4] that each countable $H_0 \subset H$ is weakly precompact and so H itself is weakly precompact. The proof is complete.

Remark 2. The theorem does not hold any more if H is a subset of $\mathbb{P}(\mu, X)$. Indeed, if $f \in \mathbb{P}(\mu, X) \backslash \mathbb{P}_c(\mu, X)$ (see [6] for an example) and satisfies (b), then f is the weak limit of the sequence $(E_{\pi_n} f)$ in $\mathbb{P}(\mu, X)$. But $C(B)$ is a (weakly) closed subspace of $\mathbb{B}(B)$ and so $f \in C(B)$, which exactly means that f is in $\mathbb{P}(\mu, X)$, so contradicting the choice of f.

Finally, we observe that using again the embedding of $\mathbb{P}_c(\mu, X)$ into $C(B)$ we can prove, without any trouble, the following result

THEOREM 2. *Let (f_n), f be in $\mathbb{P}_c(\mu, X)$. Then*

(a) *(f_n) converges weakly to f if and only if, for each $x^* \in X^*$ and each $E \in \Sigma$, the sequence $(\int_E x^* f_n(s)\, d\mu)$ converges to $\int_E x^* f(s)\, d\mu$,*

(b) *if for each $E \in \Sigma$ the sequence $(\int_E f_n(s)\, d\mu)$ is bounded in X, then (f_n) converges weakly to f if and only if for each extreme point x^* in the unit ball of X^* and each $E \in \Sigma$ the sequence $(\int_E x^* f_n(s)\, d\mu)$ converges to $\int_E x^* f(s)\, d\mu$.*

The above results were proved also by Graves and Ruess [7] in the case of vector measures with compact range, but with a more complicated proof.

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